CHAPTER 15 -- ELECTRICAL POTENTIALS

15.1)

a.) The PROTON produces an electric field which, in turn, produces an (absolute) electrical potential field. For a point mass, the electrical potential is V = kq/r, where q is the field producing charge, r is the distance between the charge and the point-of-interest, and where the sign of q must be included (i.e., a negative charge produces a negative V) in the equation. For the proton, then:

$$V_{outskirts} = (1/4\pi\varepsilon_o)q_p/r$$
$$= (9\times10^{9} \text{ volt}\cdot\text{m/C})(1.6\times10^{-19} \text{ C})/(.5\times10^{-10} \text{ m})$$
$$= 28.8 \text{ volts.}$$

b.) Electrical potential energy (that is, potential energy--U--a charge has due to its presence in an electric field) is related to electrical potential (V) by V = U/q (by definition, voltage tells you how much potential energy per unit charge exists at a point in an electric field). Including the charge's sign:

$$U_e = qV_{outskirts}$$
$$= (-1.6\times10^{-19} \text{ C})(28.8 \text{ volts})$$
$$= -4.6\times10^{-18} \text{ joules.}$$

c.) Total energy is the sum of potential and kinetic energy. For the electron:

$$E_{tot} = KE + U$$
$$= (1/2)mv^2 + qV_{outskirts}$$
$$= .5(9.1\times10^{-31} \text{ kg})(2.25\times10^{6} \text{ m/s})^2 + (-4.6\times10^{-18} \text{ joules})$$
$$= -2.3\times10^{-18} \text{ joules.}$$

Note: The negative sign suggests that the electron is in a bound state.

15.2) A positive charge will move from higher to lower electrical potential (i.e., higher to lower voltage). By definition, electric field lines do likewise. Equipotential lines will be perpendicular to the electric field lines. I've put both (answers to Part a and b) on the sketch shown on the next page.

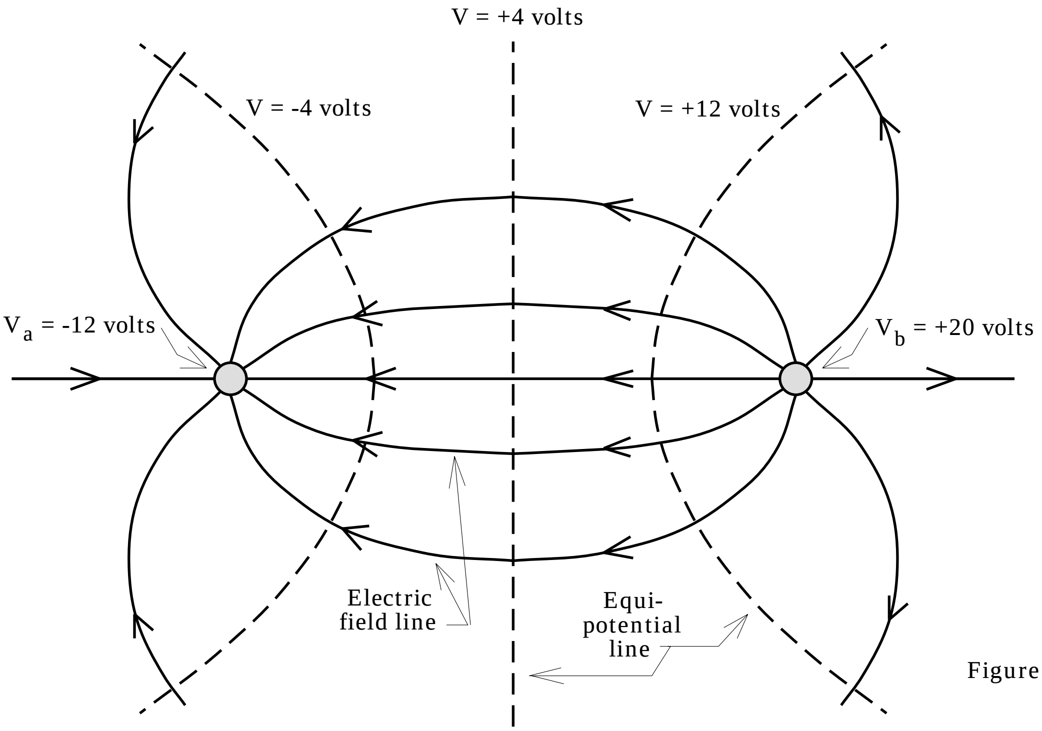


Figure I

15.3) Note that α particles are positively charged (two protons worth) and have four times the mass of a proton. When released in an electric field, an α particle will accelerate in the direction of the electric field.

a.) Assuming the electrical potential at the final point is zero volts, the electrical potential at the beginning will be $18\text{x}10^6$ volts. The relationship between the absolute electrical potential at a point and the amount of electrical potential energy a charge q has when at the point is:

$$V_1 = U_1/q_\alpha$$
$$\Rightarrow \quad U_1 = q_\alpha \qquad V_1$$
$$= [2(1.6\text{x}10^{-19}\ \text{C})][18\text{x}10^6\ \text{v}]$$
$$\Rightarrow \quad U_1 = 5.76\text{x}10^{-12}\ \text{joules.}$$

b.) But by definition, the work per unit charge is the same as the voltage difference between the two points (i.e., $W/q = -\Delta V$). That means the work per unit charge is $-(0 - 18\text{x}10^6$ volts$) = 18\text{x}10^6$ volts. Note that this is sometimes referred to as 18 Megavolts.

c.) This is a conservation of energy problem!

$$\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} = \Sigma KE_2 + \Sigma U_2$$
$$(0) + (5.76\text{x}10^{-12} \text{ j}) + (0) = (1/2)[4(1.67\text{x}10^{-27} \text{ kg})]v^2 + (0)$$
$$\Rightarrow \quad v = 4.15\text{x}10^7 \text{ m/s.}$$

15.4) The sketch below summarizes the information given in the problem.

a.) Electric field lines always proceed from higher to lower electrical potential. As such, voltage $V_A < V_B$.

b.) Because the line connecting Points D and E is perpendicular to the electric field lines, $V_D = V_E$. Knowing V_D, we can write:

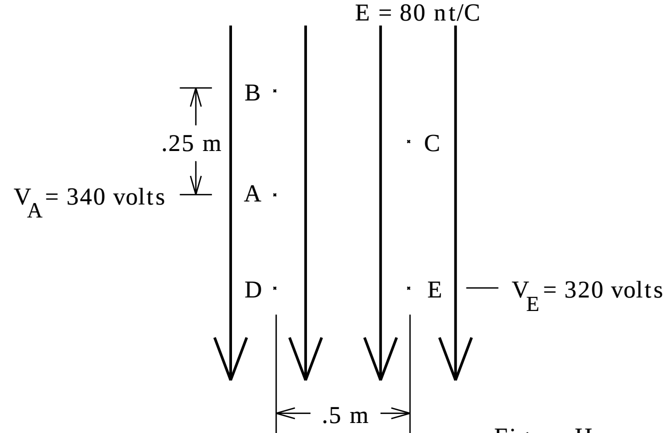


Figure II

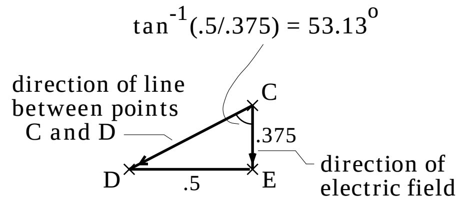
$$\Delta V = - E \cdot d$$
$$(V_D - V_A) = -E\, d_{AD} \cos 0^o$$
$$(320 \text{ v} - 340 \text{ v}) = -(80 \text{ v/m})\, d_{AD}$$
$$-20 = -80 d_{AD}$$
$$\Rightarrow \quad d_{AD} = .25 \text{ m.}$$

c.) The same relationship used in Part b can be used to determine V_B:

$$\Delta V = -E \cdot d$$
$$(V_A - V_B) = -E\, d \cos 0^o$$
$$(340 \text{ v} - V_B) = -(80 \text{ v/m})(.25 \text{ m})$$
$$\Rightarrow \quad V_B = 360 \text{ volts.}$$

d.) Determining V_C the most educationally interesting way: If the distance between Points A and B is .25 meters, and if C is half-way between those two points (vertically), and if the distance from D to A is .25

meters (calculated in Part b), then the vertical distance between Points D and C is .25 + .25/2 = .375 meters, the actual distance is $(.5^2 + .375^2)^{1/2} = .625$ meters, and the angle between a line from C to D and the vertical (i.e., in the direction of the electric field) is 53.13^o (see sketch). With all that information, we can write:



$$\Delta V = -E \cdot d$$
$$(V_D - V_C) = -E\ d \cos\phi$$
$$(320\text{ v} - V_C) = -(80\text{ v/m})(.625)\cos(53.13^o)$$
$$\Rightarrow \quad V_C = 350 \text{ volts.}$$

An alternative approach is to notice that Point C is halfway between Points A and B (at least as far as the electric field is concerned). As you know the voltages of those two points, V_c should be halfway between those two voltages.

e.) Potential energy is related to voltage at a point by $V_A = U_A/q$. This implies that:

$$U_A = qV_A$$
$$= (6\text{x}10^{-6}\text{ C})(340\text{ v})$$
$$= 2.04\text{x}10^{-3}\text{ joules.}$$

f.) Work per unit charge is related to voltage differences as $W/q = -\Delta V$. That means:

$$W/q = -(V_E - V_A)$$
$$= -(320\text{ v} - 340\text{ v})$$
$$= 20\text{ joules/C.}$$

g.) Starting with $W/q = -\Delta V$, we can write:

$$W/q = -(V_B - V_A)$$
$$\Rightarrow \quad W = -q(V_B - V_A)$$
$$= -(6\text{x}10^{-6}\text{ C})(360\text{ v} - 340\text{ v})$$
$$= -1.2\text{x}10^{-4}\text{ joules.}$$

h.) If V_A had been 340 volts and V_B had been 290 volts:

i.) Electric fields go from high to low voltages, so the electric field would have been reversed (i.e., it would have been toward the top of the page); and

ii.) The field intensity function would have been:

$$V_B - V_A = -E \cdot d_{AB}$$
$$(290 \text{ v} - 340 \text{ v}) = -E(.25 \text{ m}) \cos 0^o$$
$$\Rightarrow \quad E = 200 \text{ nt/C} \qquad \text{(or 200 volts/meter).}$$

15.5) The square is shown to the right. It will be useful to know the net electrical potential at one corner (V_{corn}) and the net electrical potential at the center (V_{cen}). As both charges are positive and equal, and as relevant distances are symmetric, the electrical potential values we need are:

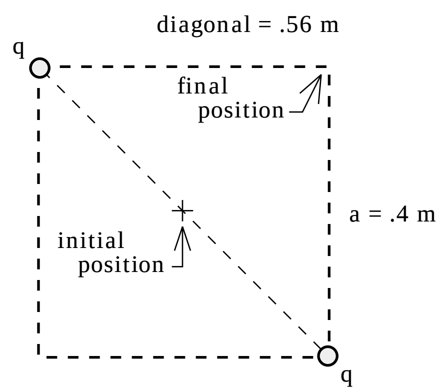


Figure III

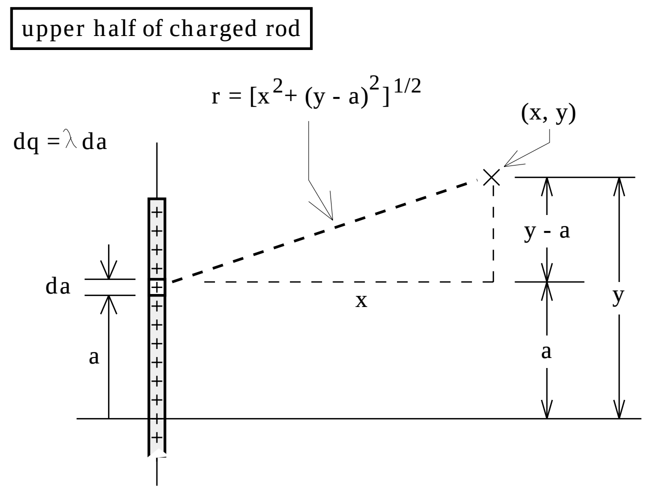
$$V_{cen} = [1/(4\pi\varepsilon_o)]q/r_{cen} + [1/(4\pi\varepsilon_o)]q/r_{cen}$$
$$= 2[(9\text{x}10^9)(10^{-16}\text{ C})/(.28)]$$
$$= 6.43\text{x}10^{-6} \text{ volts.}$$
$$V_{corn} = [1/(4\pi\varepsilon_o)]q/r_{corn} + [1/(4\pi\varepsilon_o)]q/r_{corn}$$
$$= 2[(9\text{x}10^9)(10^{-16}\text{ C})/(.4)]$$
$$= 4.5\text{x}10^{-6} \text{ volts.}$$

This is a conservation of energy problem with $U_{corn} = qV_{corn}$, etc.

$$\Sigma KE_{cen} + \Sigma U_{cen} + \Sigma W_{ext} = \Sigma KE_{corn} + \Sigma U_{corn}$$
$$(0) + q_1 V_{cen} + (0) = (1/2)mv^2 + q_1 V_{corn}$$
$$(10^{-18}\text{ C})(6.43\text{x}10^{-6}\text{ volts}) = .5(7\text{x}10^{-22}\text{ kg})v^2 + (10^{-18}\text{ C})(4.5\text{x}10^{-6}\text{ volts})$$
$$\Rightarrow \quad v = 7.43\text{x}10^{-2} \text{ m/s.}$$

15.6)

a.) This is one of those problems in which the set-up is primary while the actual evaluation of integrals is secondary. Because the geometry is

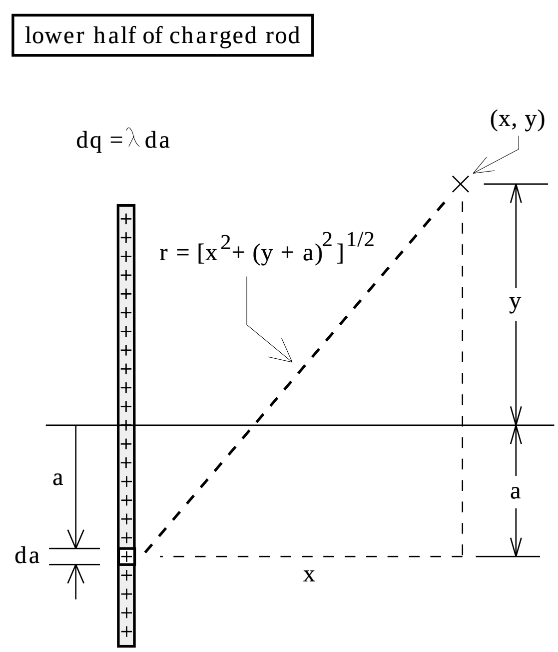


complicated and we no longer have symmetry at our disposal, we will deal with the top half of the rod, then go to the bottom half.

The figure to the right shows the situation for a differential charge dq = (λ)da on the upper part of the rod. Using that sketch, the net electrical potential for the charge from a = 0 to a = L becomes:

$$\begin{aligned} V_{top} &= \int dV \\ &= \frac{1}{4\pi\varepsilon_o}\int_{a=0}^{L}\frac{dq}{r} \\ &= \frac{1}{4\pi\varepsilon_o}\int_{a=0}^{L}\frac{\lambda da}{\left[x^2+(y-a)^2\right]^{1/2}} \\ &= \frac{\lambda}{4\pi\varepsilon_o}\int_{a=0}^{L}\frac{da}{\left[a^2+(-2y)a+(x^2+y^2)\right]^{1/2}}. \end{aligned}$$

The figure to the right shows the situation for dq on the lower part of the rod. Using that sketch and noting that a is defined as a positive number (that means the limits of integration must be from a = 0 to a = +L --see how a is used in the definition of r on the sketch), the net electrical potential for the charge on the lower half of the rod will have an expression that is very similar to the expression derived for the upper half of the rod. That expression is:



$$V_{bot} = \int dV$$
$$= \frac{\lambda}{4\pi\varepsilon_o}\int_{a=0}^{L}\frac{da}{\left(a^2+(2y)a+(x^2+y^2)\right)^{1/2}}.$$

Having set up the problem and generated the integrals to be solved, limits and all, this is as far as we really have to go. But for those stalwarts who would like to see the final solution, continue on:

The integral in both the upper rod and lower rod cases has the form:

$$\int \frac{1}{\left[k_1a^2+k_2a+k_3\right]}\,da\,.$$

The solution to an integral of this form is:

$$\frac{1}{{k_1}^{1/2}}\ln\left|2\left[k_1\left(k_1a^2+k_2a+k_3\right)\right]^{1/2}+2k_1a+k_2\right|.$$

Evaluating for the bottom half, we must substitute in $k_1 = 1$, $k_2 = 2y$, $k_3 = (x^2 + y^2)$, and solve:

$$V_{bot} = \frac{\lambda}{4\pi\varepsilon_o}\int_{a=0}^{L}\frac{1}{\left[x^2+(y+a)^2\right]^{1/2}}\,da$$
$$= \frac{\lambda}{4\pi\varepsilon_o}\left[\frac{1}{(1)^{1/2}}\ln\left|2\left(a^2+(2y)a+(x^2+y^2)\right)^{1/2}+2a+(2y)\right|\right]_{a=0}^{L}$$
$$= \frac{\lambda}{4\pi\varepsilon_o}\left[\ln\left|2\left(L^2+(2y)L+(x^2+y^2)\right)^{1/2}+2L+(2y)\right|-\ln\left|2\left(x^2+y^2\right)^{1/2}+(2y)\right|\right].$$

Assuming x and y are positive (this makes the quantity inside the absolute value positive, hence allowing us to drop the absolute value sign), the expression for the electrical potential at the point (x,y) due to the BOTTOM HALF OF THE BAR becomes:

$$V_{bot} = \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{2\left(L^2+(2y)L+(x^2+y^2)\right)^{1/2}+2L+(2y)}{2\left(x^2+y^2\right)^{1/2}+(2y)}\right].$$

For the top half, we must substitute $k_1 = 1$, $k_2 = -2y$, and $k_3 = (x^2 + y^2)$, into our general-form expression for this kind of integral. Assuming x and y are both positive (again, to allow us to drop the absolute value sign), we get:

$$V_{top} = \frac{\lambda}{4\pi\varepsilon_o} \ln\left[\frac{2\left(L^2 + (-2y)L + (x^2 + y^2)\right)^{1/2} + 2L + (-2y)}{2\left(x^2 + y^2\right)^{1/2} + (-2y)}\right].$$

The net electrical potential field at point (x, y) is the sum of V_{top} and V_{bot}.

Note: There is an interesting way to check these equations. In the text, we determined the electrical potential for this configuration specifically at x = b, y = 0. If we evaluate our expression for the bottom half, then double what we get to include the top half (the function will be symmetric at those coordinates), we get:

$$\begin{aligned} V &= 2\frac{\lambda}{4\pi\varepsilon_o} \ln\left[\frac{2\left(L^2 + b^2\right)^{1/2} + 2L}{2\left(b^2\right)^{1/2}}\right] \\ &= \frac{\lambda}{2\pi\varepsilon_o} \ln\left[\frac{\left(L^2 + b^2\right)^{1/2} + L}{b}\right]. \end{aligned}$$

Substituting in $\lambda = Q/(2L)$ gives us the same function we determined in the text.

b.) With the electrical potential function V, we could use minus the del operator to determine E.

15.7) Using the del operator, we get:

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\left[\frac{\partial\left[k_1 e^{-kx} + k_2 / y^3\right]}{\partial x}\mathbf{i} + \frac{\partial\left[k_1 e^{-kx} + k_2 / y^3\right]}{\partial y}\mathbf{j} + \frac{\partial\left[k_1 e^{-kx} + k_2 / y^3\right]}{\partial z}\mathbf{k}\right] \\ &= -\left[\left(-kk_1 e^{-kx}\right)\mathbf{i} + \left(k_2 \frac{-3}{y^4}\right)\mathbf{j}\right]. \end{aligned}$$

15.8)

a.) The electric potential should be zero where the electric field is zero (if there is such a place). In this case, that will occur at x = ∞, y = ∞.

b.) Using those coordinates for our zero electrical potential point, we can write:

$$
\begin{aligned}
V(x,y) - V(\infty,\infty) &= -\int \mathbf{E} \bullet d\mathbf{r} \\
&= -\int_{x=\infty,y=\infty}^{x,y} \left[k_1 e^{-kx}\mathbf{i} + (k_2/y^3)\mathbf{j}\right] \bullet \left[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right] \\
&= -\int_{x=\infty,y=\infty}^{x,y} \left[k_1 e^{-kx}dx + (k_2/y^3)dy\right] \\
&= -\int_{x=\infty}^{x} \left[k_1 e^{-kx}dx\right] - \int_{y=\infty}^{y} \left[(k_2 y^{-3})dy\right]
\end{aligned}
$$

$$\Rightarrow \quad V(x,y) = \left[\frac{k_1}{k} e^{-kx}\right]_{x=\infty}^{x} + \left[\frac{k_2}{2y^2}\right]_{y=\infty}^{y}$$

$$\Rightarrow \quad V(x,y) = \left[\frac{k_1}{k} e^{-kx} - 0\right] + \left[\frac{k_2}{2y^2} - 0\right].$$

15.9)

a.) The electric field close to the surface of a large conducting sheet is constant and perpendicular to the sheet's face. As equipotential lines (and surfaces) are perpendicular to electric field lines, the surfaces themselves will be parallel to the sheet's face. As the electric field is a constant, the spacing between surfaces at regular voltage intervals will be uniform.

b.) The electric field close to a conducting surface is σ/ε_o (this was derived in the text). The relationship between a constant electric field and an electrical potential difference within the field is:

$$\Delta V = -\mathbf{E} \bullet \mathbf{d},$$

where ΔV is the voltage difference over a distance d. Remembering that a move from higher to lower electrical potential produces an electrical potential difference that is negative, we find that in this case ΔV = - 12 volts.

With this, we can determine the distance d over which ΔV = -12 volts;

$$\Delta V = -\mathbf{E} \bullet \mathbf{d}$$

$$\Rightarrow \quad (-12 \text{ volts}) = -\left(\frac{\sigma}{\varepsilon_o}\right)(d)\left(\cos 0^o\right)$$

$$\Rightarrow \quad d = (12 \text{ volts})\left(\frac{8.85x10^{-12} C/v \bullet m}{10^{-10} C/m^2}\right)$$

$$= 1.062 \text{ meters.}$$

15.10) The relationship between a known electric field E and its associated electrical potential function is:

$$\Delta V = -\int \mathbf{E} \bullet d\mathbf{r}.$$

From this, we can see that an electrical potential difference, hence electrical potential function, is related to the area under the electric field versus position plot. As areas change continuously, an electrical potential function must be continuous.

The relationship between a known electrical potential function V and its associated electric field E is:

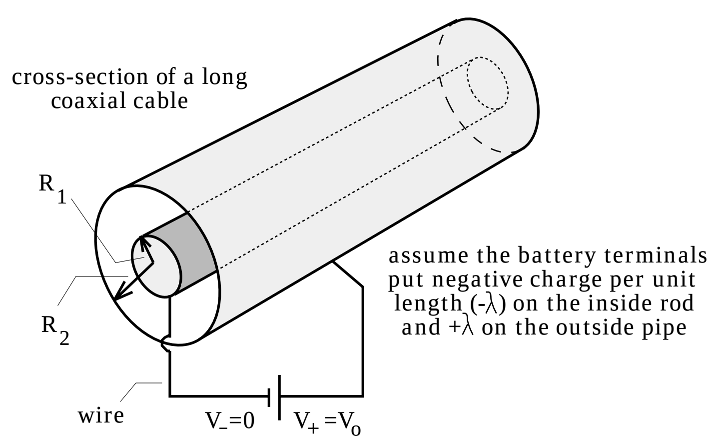
$$\mathbf{E} = -\nabla V.$$

From this, we can see an electric field is related to the slope of its electrical potential versus position plot. As electrical potential functions can change abruptly, their slopes are not always continuous (hence electric field functions do not have to be continuous). This shouldn't be surprising as any force related function can be discontinuous from point to point.

15.11) The sketch to the right shows the set-up. We need to determine the electric field between the plates before we can use:

$$\Delta V = -\int \mathbf{E} \bullet d\mathbf{r}.$$

Noticing that negative charge will accumulate on the inner rod with the opposite of that charge on the outer wall, and assuming a cylindrical Gaussian surface of length L and radius r (between R_1 and R_2), we can write:



$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi r L) = \frac{-\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{-\lambda}{2\pi\varepsilon_o r}.$$

Electric fields go from positive to negative charges. The inner rod (radius R_1) has negative charge on it and has an electrical potential of zero due to its connection with the battery. That means that as we proceed from the inner rod outward, we move <u>against</u> the electric field set up in that region. With this information, and noting that r is a unit vector directed away from the central rod:

$$V(r) - V(R_1) = -\int_{r=R_1}^{r} \mathbf{E} \bullet d\mathbf{r}$$

$$\Rightarrow V(r) - (0) = -\int_{r=R_1}^{r} \left[\frac{\lambda}{2\pi\varepsilon_o r}(-\mathbf{r}) \right] \bullet [dr\mathbf{r}]$$

$$\Rightarrow \quad V(r) = -\int_{r=R_1}^{r} \left(\frac{\lambda}{2\pi\varepsilon_o} \frac{1}{r} \right) dr \left(\cos 180^o \right)$$

$$\Rightarrow \quad V(r) = \frac{\lambda}{2\pi\varepsilon_o} \int_{r=R_1}^{r} \left(\frac{1}{r} \right) dr$$

$$\Rightarrow \quad V(r) = \frac{\lambda}{2\pi\varepsilon_o} \ln\left(\frac{r}{R_1} \right).$$

Does this expression make sense? One way to check the expression is to see what it predicts for a voltage at a point we know. At R_1 on the inside rod, the electrical potential is supposed to be zero. Putting $r = R_1$ into our expression, we get $\ln(R_1/R_1)$. This equals zero--the defined electrical potential for the rod. So far, so good.

Another way to check our expression is to consider a test charge put in the field. That is, if a positive test charge is placed at the outer pipe (i.e., at R_{big}) and we allow it to accelerate toward the central rod (i.e., to R_{small}), the work done by the field should be positive. Does our derived electrical potential function predict that? (The work should be positive).

Doing the calculation yields:

$$W = -q[\Delta V]$$
$$= -q\left[\frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{R_{small}}{R_1}\right) - \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{R_{big}}{R_1}\right)\right].$$

The right-hand quantity in the brackets is larger, which means the work done will be positive as expected.

15.12) The 2Q's worth of free charge will move to the outside of the conductor. In addition to that, -Q's worth of charge will migrate to the inside of the conductor in response to the presence of the +Q's worth of charge hanging at the center of the configuration, leaving an additional +Q's worth of charge on the outer surface of the conductor. Summarily, there will be +3Q's worth of charge on the outside surface, -Q's worth of charge on the inside surface, and +Q's worth of charge hanging at the center. We can use Gauss's Law to determine the electric fields in the various regions. Without showing the work, they are:

For $r < R_1$: $$E = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r^2}.$$

Inside the conductor $$E = 0.$$

For $r > R_2$: $$E = \frac{3Q}{4\pi\varepsilon_o}\frac{1}{r^2}.$$

Assuming the electrical potential is zero at infinity, the electrical potential functions are as follows:

For $r > R_2$:

$$V(r) - V(\infty) = -\int_{r=\infty}^{r} \mathbf{E} \bullet d\mathbf{r}$$
$$\Rightarrow \quad V(r) = -\int_{r=\infty}^{r}\left[\frac{3Q}{4\pi\varepsilon_o}\frac{1}{r^2}\mathbf{r}\right] \bullet dr\mathbf{r}$$
$$\Rightarrow \quad V(r) = \frac{3Q}{4\pi\varepsilon_o}\frac{1}{r}.$$

Inside the conductor:

$$\left[V(R_2)-V(\infty)\right]+\left[V(r)-V(R_2)\right]=-\int_{r=\infty}^{R_2}\mathbf{E}_{out}\bullet d\mathbf{r}-\int_{r=R_2}^{r}\mathbf{E}_{cond}\bullet d\mathbf{r}$$

$$\Rightarrow \quad V(r)=-\int_{r=\infty}^{R_2}\left[\frac{3Q}{4\pi\varepsilon_o}\frac{1}{r^2}\mathbf{r}\right]\bullet dr\mathbf{r}-\int_{r=R_2}^{r}[0]\bullet dr\mathbf{r}$$

$$\Rightarrow \quad V(r)=\frac{3Q}{4\pi\varepsilon_o}\frac{1}{R_2}$$ (note that this is a **constant**).

For $r < R_1$:

$$\left[V(R_2)-V(\infty)\right]+\left[V(R_1)-V(R_2)\right]+\left[V(r)-V(R_1)\right]=-\int_{r=\infty}^{R_2}\mathbf{E}_{out}\bullet d\mathbf{r}-\int_{r=R_2}^{R_1}\mathbf{E}_{cond}\bullet d\mathbf{r}-\int_{r=R_1}^{r}\mathbf{E}_{in}\bullet d\mathbf{r}$$

$$\Rightarrow \quad V(r)=-\int_{r=\infty}^{R_2}\left[\frac{3Q}{4\pi\varepsilon_o}\frac{1}{r^2}\mathbf{r}\right]\bullet dr\mathbf{r}-\int_{r=R_2}^{R_1}[0]\bullet dr\mathbf{r}-\int_{r=R_1}^{r}\left[\frac{Q}{4\pi\varepsilon_o}\frac{1}{r^2}\mathbf{r}\right]\bullet dr\mathbf{r}$$

$$\Rightarrow \quad V(r)=\frac{3Q}{4\pi\varepsilon_o}\frac{1}{R_2}+\frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r}-\frac{1}{R_1}\right]$$

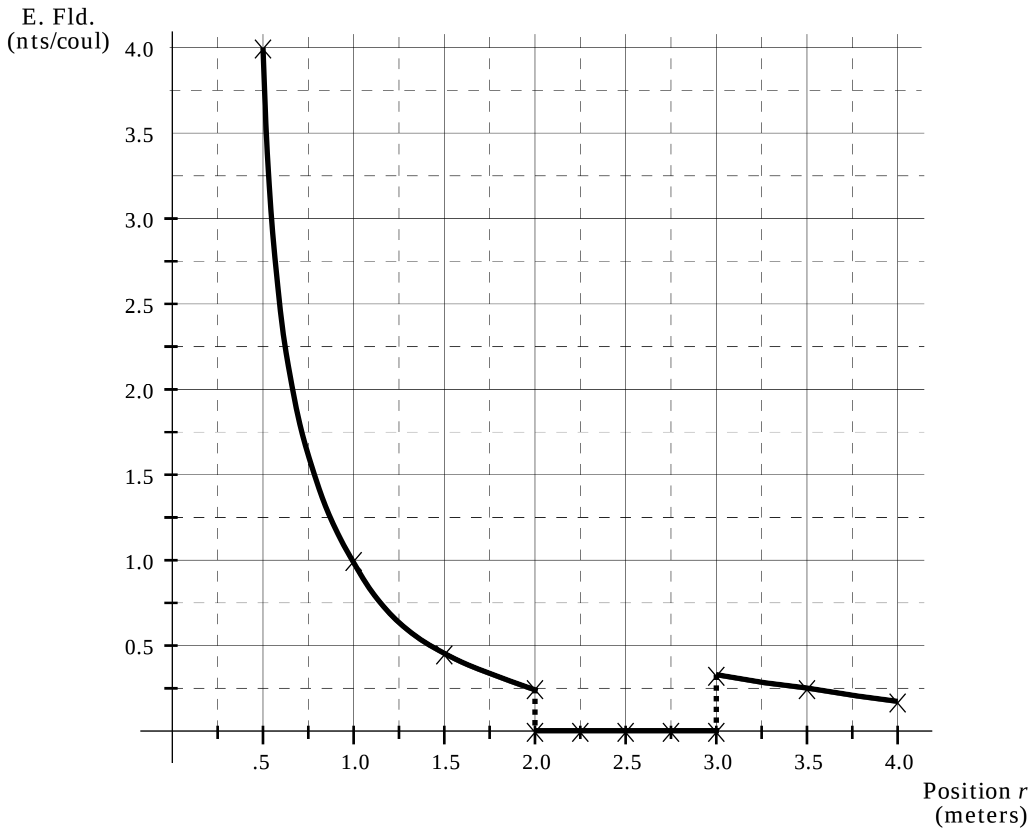
$$\Rightarrow \quad V(r)=\frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r}-\frac{1}{R_1}+\frac{3}{R_2}\right].$$

Summarizing our information:

range	electric field	electrical potential
for $r < R_1$	$E=\frac{Q}{4\pi\varepsilon_o}\frac{1}{r^2}$	$V(r)=\frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r}-\frac{1}{R_1}+\frac{3}{R_2}\right]$
inside the conductor	$E=0$	$V(r)=\frac{3Q}{4\pi\varepsilon_o R_2}$
for $r > R_2$	$E=\frac{3Q}{4\pi\varepsilon_o}\frac{1}{r^2}$	$V(r)=\frac{3Q}{4\pi\varepsilon_o r}$

The graph of this information is shown on the next page.

Electric Field versus Position Plot



Electrical Potential versus Position Plot